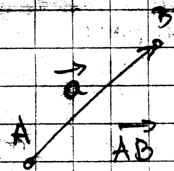
OSNOVNI POJMOVI VEKTORSKE ALGEBRE

Vektori imaju intenzitet, pravac i smjer.



$\vec{a} + (-\vec{a}) = \vec{0}$

Jedinični vektor je vektor: $\vec{a}^0 = \frac{1}{|\vec{a}|} \cdot \vec{a}$

Vektori $\vec{a}_1, \dots, \vec{a}_n$ su linearno zavisni ako postoje skalari $\lambda_1, \dots, \lambda_n$ od kojih je bar jedan različit od nule, tako da vrijedi:

$$\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \dots + \lambda_n \vec{a}_n = 0$$

A lin. nezavisni, ako je $\lambda_1 = \lambda_2 = \dots = \lambda_n$

Specijalni slučaj afinih koor. jesu pravougle dekartove koor.

$\vec{i}, \vec{j}, \vec{k}$ – jedinični vektori koji su ortogonalni

$\vec{a}(a_x, a_y, a_z)$

$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$ $\quad 0 \le \varphi \le \pi$ skalarni proizvod

$|\vec{a}| = \sqrt{\vec{a}^2} = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$ vektorski proizvod

$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

1. Odrediti ugao koji zahvataju vektori:

$\vec{a} = 3\vec{p} + 2\vec{q}$

$\vec{b} = \vec{p} + 5\vec{q}$

gdje su $\vec{p}$ i $\vec{q}$ uzajamno normalni ortovi

$$\cos\varphi = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

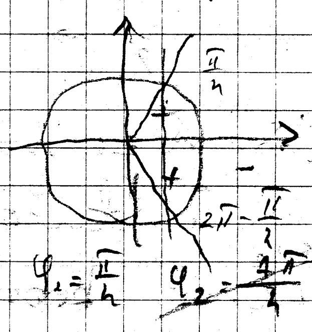
$$\vec{a}\cdot\vec{b} = (3\vec{p} + 2\vec{q})\cdot(\vec{p} + 5\vec{q}) = 3\vec{p}^2 + 10\vec{q}^2 = 3\cdot 1 + 10\cdot 1 = 13$$

$$|\vec{a}| = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$$

$$|\vec{b}| = \sqrt{1+25} = \sqrt{26}$$

$$\cos\varphi = \frac{13}{\sqrt{13}\cdot\sqrt{26}} = \frac{13}{\sqrt{13\cdot 2\cdot 13}} = \frac{\cancel{13}}{\cancel{13}\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad 0 \le \varphi \le \pi$$

$$\varphi = \frac{\pi}{4}$$



2. Odrediti parametar λ tako da intenziteti vektora: $\vec{a} = \{2e^{\lambda}, \lambda, \lambda+1\}$, $\vec{b} = \{\lambda+1, \lambda-2, 0\}$ budu jednaki a zatim naći ugao između njih.

$$|\vec{a}| = \sqrt{4e^{2\lambda} + \lambda^2 + \lambda^2 - 2\lambda + 1} = \sqrt{4e^{2\lambda} + 2\lambda^2 - 2\lambda + 1}$$

$$|\vec{b}| = \sqrt{\lambda^2 + 2\lambda + 1 + \lambda^2 - 4\lambda + 4 + 0} = \sqrt{2\lambda^2 - 2\lambda + 5}$$

$$\sqrt{4e^{2\lambda} + 2\lambda^2 - 2\lambda + 1} = \sqrt{2\lambda^2 - 2\lambda + 5} \quad /^2$$

$$4e^{2\lambda} + \cancel{2\lambda^2} - \cancel{2\lambda} + 1 = \cancel{2\lambda^2} - \cancel{2\lambda} + 5$$

$$4e^{2\lambda} = 4 \quad /:4 \qquad e^{2\lambda} = 1 \Rightarrow e^{2\lambda} = e^0 \qquad \lambda = 0$$

$\vec{a} = \{2, 0, -1\}$ $\vec{b} = \{1, -2, 0\}$ $|\vec{a}| = \sqrt{4+1} = \sqrt{5}$ $|\vec{b}| = \sqrt{5}$

$\cos\varphi = \frac{2+0}{\sqrt{5}\cdot\sqrt{5}} = \frac{2}{5} \Rightarrow \varphi = \arccos\frac{2}{5}$

3. Izračunati dužinu vektora $\vec{a} = (3\vec{i} + 4\vec{j} + 5\vec{k}) \times (\vec{i} + 6\vec{j} + 4\vec{k})$.

$$D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ 1 & 6 & 4 \end{vmatrix} = \vec{i}(16-30) - \vec{j}(12-5) + \vec{k}(18-4)$$

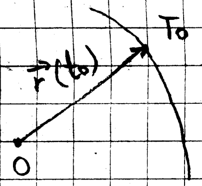
$$= -14\vec{i} - 7\vec{j} + 14\vec{k}$$

$$|\vec{a}| = \sqrt{14^2 + 7^2 + 14^2} = \sqrt{2^2\cdot 7^2 + 7^2 + 2^2\cdot 7^2} = \sqrt{7^2(4+1+4)} = 7\cdot\sqrt{9} = 7\cdot 3 = 21$$

VEKTORSKE F-JE SKALARNOG ARGUMENTA

- Ako je S neprazan podskup skupa realnih br $\mathbb{R}$, onda svako preslikavanje skupa S u vektorski prostor X_0 radij vektora, tako da svakom elementu $t \in S$ bude pridružen vektor $\vec{u}(t) \in X_0$ zove vektorska f-ja skalarnog argumenta u oznaci $\vec{u} = \vec{u}(t)$, odnosno: $\vec{u}(t) = u_x(t)\vec{i} + u_y(t)\vec{j} + u_z(t)\vec{k}$
F-je u_x, u_y i u_z predstavljaju realne f-je koje se zovu koordinatne f-je vektorske f-je $\vec{u}(t)$.
Derivacija vektorske f-je je nova vektorska f-ja $\vec{u}'(t) = \frac{d\vec{u}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}$.

- Neka je $\vec{r} = \vec{r}(t)$ čija je vektorska f-ja definisana na otvorenom intervalu I realnih brojeva. Tada je svakom realnom br. $t_0 \in I$ pridružen vektor $\vec{r}(t_0)$ i taj je vektor radij vektor neke tačke T_0. Skup svih tačaka T_0 za koje je $t_0 \in I$ zovemo HODOGRAFOM zadane vektor fje.



Ako je vektorska fja $\vec{r}(t)$ diferencijabilna onda taj hodograf zovemo krivom ili krivuljom.

1. Data je kriva $\alpha: R \to E^3$ koja je zadana svojom vektorskom jed. a) $\vec{r} = u\vec{i} + u^2\vec{j}$

a) $x = u$, $y = u^2$, $z = 0$

$y = x^2$ $z = 0$ parabola

b) $\vec{r} = t\vec{i} + t\vec{j}$

$x = t$ $y = t$ $z = 0$

$y = x$ prava

c) $\alpha: [0, 2\pi] \to E^3$

$\vec{r}(t) = a\cos t\,\vec{i} + b\sin t\,\vec{j}$ $t \in [0, 2\pi]$ $a > 0, b > 0$

$x = a\cos t$ $y = b\sin t$ $z = 0$

$\cos t = \frac{x}{a}$ $\sin t = \frac{y}{b}$ $/^2$ cilj je eliminisati t

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ elipsa

2. Dokazati da grafik zatvorene krive koja je zadana

$\alpha: [0,\pi] \to E^3$: $x = \sin 2\varphi$ $y = 1-\cos 2\varphi$ $z = 2\cos\varphi$

leži na sferi i da je presjek paraboličkog i kružnog valjka "obični"

$$x^2+y^2+z^2 = \sin^2 2\varphi + (1-\cos 2\varphi)^2 + 4\cos^2\varphi$$

$$= \sin^2 2\varphi + 1 - 2\cos 2\varphi + \cos^2 2\varphi + 4\cos^2\varphi$$

$$= \sin^2 2\varphi + 1 - 2(\cos^2\varphi - \sin^2\varphi) + \cos^2 2\varphi + 4\cos^2\varphi$$

$$= 2 - 2\cos^2\varphi + 2\sin^2\varphi + 4\cos^2\varphi =$$

$$= 2 + 2(\cos^2\varphi + \sin^2\varphi) = 4 \quad \text{konačan br.} \Rightarrow \text{sfera}$$

$z^2 = -2y + 4$ $\quad -2y = z^2 - 4$ $\quad y = \frac{-z^2}{2} - \frac{4}{2}$ $\quad \frac{y}{2} + \frac{z^2}{2^2} = 1$

$$x^2 + (y-1)^2 = \sin^2 2\varphi + \cos^2 2\varphi = 1$$

$$\frac{y}{2} + \left(\frac{z}{2}\right)^2 = \sin^2\varphi + \cos^2\varphi = 1$$

3. Naći dužinu luka krive $\alpha: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to E$ zadane:

$\vec{r} = \{\sin^2 t, \sin t\cos t, \ln\cos t\}$ $\quad t=0,\ t=t$ granice

$s = ?$

$x = \sin^2 t$ $\quad y = \sin t\cos t$ $\quad z = \ln\cos t$

$\dot{x} = 2\sin t\cos t = \sin 2t$

$\dot{y} = \cos^2 t - \sin^2 t = \cos 2t$

$\dot{z} = \frac{1}{\cos t}(-\sin t) = -\operatorname{tg} t$

$$s = \int_0^t \sqrt{\sin^2 2t + \cos^2 2t + \operatorname{tg}^2 t}\, dt = \int_0^t \sqrt{1+\operatorname{tg}^2 t}\, dt = \int_0^t \frac{dt}{\cos t} = \frac{1}{\alpha}\ln\operatorname{tg}\left(\frac{\alpha x}{2}+\ldots\right)$$

$$= \ln\operatorname{tg}\left(\frac{t}{2}+\frac{\pi}{4}\right)\Big|_0^t = \ln\operatorname{tg}\left(\frac{t}{2}+\frac{\pi}{4}\right)$$

4. Naći dužinu luka $x^2 = 3y$, $2xy = 9z$ od neke tačke $(0,0,0)$ do $(3,3,2)$

$x = 3t \quad y = 3t^2 \quad z = \frac{2\cdot 3t\cdot 3t^2}{9} = 2t^3$

$\dot{x} = 3$

$\dot{y} = 6t$

$\dot{z} = 6t^2$

granice: $x = 3t$ u 0 i 3:

$x = 3t \quad 3 = 3t$

$0 = 3t \quad t = 1$

$t = 0$

$$S = \int_0^1 \sqrt{9 + 36t^2 + 36t^4}\,dt = \int_0^1 \sqrt{9(1+4t^2+4t^4)}\,dt = 3\int_0^1 \sqrt{1+4t^2+4t^4}\,dt =$$

$$= 3\cdot\int_0^1 \sqrt{(2t^2+1)^2}\,dt = 3\cdot\int_0^1 (2t^2+1)\,dt = 3\cdot\left(2\cdot\frac{t^3}{3} + t\right)\Big|_0^1 =$$

$$= 3\cdot\left(\frac{2}{3}\cdot 1 + 1\right) = 2 + 3 = 5$$

5. Pokazati da grafik krive $\alpha : R \to E^3$ zadane:

$x = \frac{t}{1+t^2+t^4} \quad y = \frac{t^2}{1+t^2+t^4} \quad z = \frac{t^3}{1+t^2+t^4}$ leži na kugli sa središtem u tački $S(0, \frac{1}{2}, 0)$ poluprečnika $\frac{1}{2}$.

$$x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{t^2}{(1+t^2+t^4)^2} + \frac{t^4}{(1+t^2+t^4)^2} - \frac{t^2}{1+t^2+t^4} + \frac{1}{4} + \frac{t^6}{(1+t^2+t^4)^2}$$

$$= \frac{\cancel{t^2} + \cancel{t^4} - \cancel{t^2} - \cancel{t^4} - \cancel{t^6} + \cancel{t^6}}{(1+t^2+t^4)^2} + \frac{1}{4} = \frac{1}{4}$$

6. Naći dužinu luka krive $\alpha: \mathbb{R} \to E^3$ zadane sa:

$x = a(t - \sin t) \qquad y = a(1-\cos t) \qquad z = 4a \cos\frac{t}{2}$

između njena 2 sjecišta sa xOz. $y=0$

$x = at - a\sin t \qquad \dot{x} = a - a\cos t$

$y = a - a\cos t \qquad \dot{y} = a \sin t$

$z = 4a \cdot \cos\frac{t}{2} \qquad \dot{z} = -4a \cdot \sin\frac{t}{2} \cdot \frac{1}{2} = -2a \sin\frac{t}{2}$

$a(1-\cos t) = 0 \Rightarrow -\cos t = -1 \Rightarrow \cos t = 1 \Rightarrow t = 0, 2\pi$

$$S = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t + 4a^2 \sin^2\frac{t}{2}}\, dt =$$

$$= \int_0^{2\pi} a\sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1} + 4a^2 \sin^2\frac{t}{2}}\, dt =$$

$$= \int_0^{2\pi} a\sqrt{2 - 2\cos t + 44\sin^2\frac{t}{2}}\, dt = \qquad \left(\frac{1-\cos\alpha}{2}\right)$$

$$= \int_0^{2\pi} a\sqrt{2 - 2\cos t + \frac{4}{2}\,\frac{1-\cos t}{2}}\, dt = \qquad 8a\sqrt{2}$$

$$= \int_0^{2\pi} a\sqrt{2 - 2\cos t + 2 - 2\cos t)}\, dt =$$

$$= a \cdot \int_0^{2\pi} \sqrt{4(1-\cos t)}\, dt = \qquad 2\sin^2\frac{t}{2}$$

$$= 2a\int_0^{2\pi} \sqrt{2\sin^2\frac{t}{2}}\, dt = 2a\sqrt{2}\int_0^{2\pi} \sin\frac{t}{2}\, dt =$$

$$= 2a\sqrt{2}\left(-\cos\frac{t}{2}\cdot 2\right)\Big|_0^{2\pi} = 4a\sqrt{2}\,(\underbrace{\underbrace{-\cos\pi}_{1} + \underbrace{\cos 0}_{1}}_{2}) =$$

$= 8a\sqrt{2}$

7. Naći dužinu luka krive $x^3 = 3a^2 y$ $2xz = a^2$ među ravnima $y = \frac{a}{3}$, $y = 9a$. $s = 9a$ parametrizaciju

8. Pokazati da zatvorena kriva $\alpha : [0, \frac{\pi}{2}] \to E^3$
$x = \cos^3 t$ $y = \sin^3 t$, $z = \cos 2t$ i ~~$s = 15$~~ $s = \frac{5}{2}$

7. $x^3 = 3a^2 y$ $2xz = a^2$

$x = at$

$a^3 t^3 = 3a^2 y \Rightarrow at^3 = 3y \Rightarrow y = \frac{at^3}{3}$

$2 \cdot at\, z = a^2 \Rightarrow z = \frac{a}{2t}$

$x = t$ $y = \frac{t^3}{3a^2}$ $z = \frac{a^2}{2t}$ $(t^{-1})' = -1 t^{-2}$

$\dot{x} = 1$ $\dot{y} = \frac{3t^2}{3a^2}$ $\dot{z} = \frac{-a^2}{2t^2}$

$\frac{t^3}{3a^2} = \frac{a}{3}$ $3t^3 = 3a^3$ $t = a$

$\frac{t^3}{3a^2} = 9a$ $t^3 = 27a^3$ $t = 3a$

$$s = \int_a^{3a} \sqrt{1 + \frac{t^4}{a^4} + \frac{a^4}{4t^4}}\, dt = \int_a^{3a} \sqrt{\frac{4a^4 t^4 + 4t^8 + a^8}{a^4 \cdot 4t^4}}\, dt$$

$$= \int_a^{3a} \sqrt{\frac{(2t^4 + a^4)^2}{(2a^2t^2)^2}}\, dt = \int_a^{3a} \frac{2t^4 + a^4}{2a^2t^2}\, dt = \int_a^{3a} \frac{2t^{4\,2}}{2a^2 t^2}\, dt + \int_a^{3a} \frac{a^{4\,2}}{2a^2 t^2}\, dt =$$

$$= \frac{1}{a^2}\int_a^{3a} t^2\, dt + \frac{a^2}{2}\int_a^{3a} t^{-2}\, dt = \frac{1}{a^2} \cdot \frac{t^3}{3}\Big|_a^{3a} + \frac{a^2}{2} \cdot \frac{t^{-1}}{-1}\Big|_a^{3a} =$$

$$= \frac{1}{a^2} \cdot \left(\frac{27a^3}{3} - \frac{a^3}{3}\right) - \frac{a^2}{2} \cdot \left(\frac{1}{3a} - \frac{1}{a}\right) = \frac{1}{a^2} \cdot \frac{26a^3}{3} + \frac{a^2}{2} \cdot \frac{2}{3a} =$$

$$= \frac{26a}{3} + \frac{a}{3} = 9a$$

8. $x=\cos^3 t,\quad y=\sin^3 t,\quad z=\cos 2t$

$\dot{x}=-3\cos^2 t\cdot\sin t$

$\dot{y}=3\sin^2 t\cdot\cos t$

$\dot{z}=(\cos^2 t-\sin^2 t)'_t=-2\cos t\sin t-2\sin t\cos t=-4\sin t\cos t$

$$s=\int_0^{\frac{\pi}{2}}\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cdot\cos^2 t+16\sin^2 t\cos^2 t}\,dt=$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{\cos^2 t\sin^2 t\,(9\underbrace{(\cos^2 t+\sin^2 t)}_{1}+16)}\,dt=$$

$$=\int_0^{\frac{\pi}{2}}5\sin t\cos t\,dt=\left|\begin{array}{ll}\sin t=z & t=0\Rightarrow z=0\\ \cos t\,dt=dz & t=\frac{\pi}{2}\quad z=1\end{array}\right|=$$

$$=5\cdot\int_0^1 z\,dz=5\cdot\frac{z^2}{2}\Big|_0^1=\frac{5}{2}(1-0)=\frac{5}{2}$$

1. Naći one tangente krive $\alpha : \mathbb{R} \to E^3$ zadane vektorski (t, t^2, t^3) koje su paralelne s ravni $x+2y+z-3=0$

prvo moramo naći tačku dodira, pa:

Vektor tangente (smjera) u skalarnom proizvodu s vektorom normale zadane ravni dat će nam vrijednost parametra t na osnovu kojeg ćemo pronaći tačke dodira.

izvodi:

$t' = 1$

$(t^2)' = 2t$

$(t^3)' = 3t^2$

$\vec{N} = (1, 2, 1)$

$\vec{r}' = (1, 2t, 3t^2)$

$\vec{N} \cdot \vec{r}' = 0$

$1 + 4t + 3t^2 = 0$

$t_{1,2} = \frac{-4 \pm \sqrt{16-12}}{6} = \frac{-4 \pm 2}{6} \begin{cases} -1 \\ -\frac{1}{3} \end{cases}$

$\vec{r} = \{t, t^2, t^3)$

$D_1 : \left(-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}\right)$

$D_2 : (-1, 1, -1)$

$t_1 : \frac{x+\frac{1}{3}}{1} = \frac{y-\frac{1}{9}}{-\frac{2}{3}} = \frac{z+\frac{1}{27}}{\frac{1}{3}} \longrightarrow \vec{r}'\left(\frac{1}{3}\right)$

$t_2 : \frac{x+1}{1} = \frac{y-1}{-2} = \frac{z+1}{3} \longrightarrow r'(-1)$

2. Dokazati da tangente krive $\alpha: \mathbb{R} \to E^3$, zadane $x = a(\sin t + \cos t)$, $y = a(\sin t - \cos t)$, $z = be^{-t}$, sijeku ravan xOy u kružnici $x^2 + y^2 = 4a^2$.

$$\dot{x} = (a\sin t + a\cos t)' = a\cos t - a\sin t = a(\cos t - \sin t)$$

$$\dot{y} = a(\cos t + \sin t)$$

$$\dot{z} = -be^{-t}$$

$$\frac{x - a(\sin t + \cos t)}{a(\cos t - \sin t)} = \frac{y - a(\sin t - \cos t)}{a(\cos t + \sin t)} = \frac{z - be^{-t}}{-be^{-t}} = \lambda$$

$$x = a(\sin t + \cos t) + \lambda a(\cos t - \sin t)$$

$$y = a(\sin t - \cos t) + \lambda a(\cos t + \sin t)$$

$$z = be^{-t} - \lambda be^{-t}$$

$$z = 0 \Rightarrow be^{-t} - \lambda be^{-t} = 0$$

$$\underbrace{be^{-t}}_{\neq 0 \text{ jer je kriva}}(1-\lambda) = 0 \Rightarrow 1 - \lambda = 0 \Rightarrow \lambda = 1$$

$$x = \cancel{a\sin t} + a\cos t + a\cos t - \cancel{a\sin t} = 2a\cos t$$

$$y = a\sin t - \cancel{a\cos t} + \cancel{a\cos t} + a\sin t = 2a\sin t$$

$$x^2 + y^2 = 4a^2\cos^2 t + 4a^2\sin^2 t = 4a^2(\underbrace{\cos^2 t + \sin^2 t}_{1})$$

$$x^2 + y^2 = 4a^2 \quad \text{q.e.d.}$$

3. Napisati jed. tangente krive $x=e^t\cos t$, $y=e^t\sin t$, $z=e^t$ za $t=0$.

prvo tražimo jed. tang. pa uvrstimo $t=0$

$\dot{x} = e^t\cdot\cos t - e^t\sin t$

$\dot{y} = e^t\cdot\sin t + e^t\cos t$

$\dot{z} = e^t.$

$$\frac{x-e^t\cos t}{e^t\cos t - e^t\sin t} = \frac{y-e^t\sin t}{e^t\sin t + e^t\cos t} = \frac{z-e^t}{e^t} \quad t=0 \Rightarrow$$

$$\frac{x-1}{1-0} = \frac{y-0}{0+1} = \frac{z-1}{1} \Rightarrow \frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{1}$$

4. Napisati jed. tangente krive $x^2+y^2=10$, $y^2+z^2=25$ u tački $T(1,3,4)$

$$(*)\quad t: \frac{x-1}{1} = \frac{y-3}{-\frac{1}{3}} = \frac{z-4}{\frac{1}{4}}$$

Uzet ćemo da je parametar apscisa x pa naša jed. je:

$\vec{r}(x) = x\vec{i} + y(x)\vec{j} + z(x)\vec{k}$

ali mi nećemo izražavati x nego radimo izvod:

$$\begin{cases} 2x + 2y\dot{y} = 0 \\ 2y\dot{y} + 2z\dot{z} = 0 \end{cases}$$

$2x - 2z\dot{z} = 0$

$2\cdot 1 - 2\cdot 4\cdot\dot{z} = 0$

$2 - 8\dot{z} = 0$

$\boxed{\dot{z} = \frac{1}{4}}$ $\boxed{\dot{x} = 1}$ jer je po apscisi

$2x + 2y\dot{y} = 0$

$2\cdot 1 + 2\cdot 3\dot{y} = 0$

$6\dot{y} = -2$

$\boxed{\dot{y} = -\frac{1}{3}}$

pa uvrstimo u $(*)$

5. U kojim je tačkama tangenta na krivu $\alpha: \mathbb{R} \to E^3$
$x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$ paralelna s ravni:
$3x + y + z + 2 = 0$

$\dot{x} = 3 - 3t^2$

$\dot{y} = 6t$

$\dot{z} = 3 + 3t^2$

$\vec{N}(3, 1, 1)$

$\vec{r}'(3 - 3t^2, 6t, 3 + 3t^2)$

$\vec{N} \cdot \vec{r}' = 0$

$9 - 9t^2 + 6t + 3 + 3t^2 = 0$

$-6t^2 + 6t + 12 = 0 \quad /:6$

$-t^2 + t + 2 = 0$

$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{-2} = \frac{-1 \pm 3}{-2} < \begin{matrix} 2 \\ -1 \end{matrix}$

$\vec{r}(3t - t^3, 3t^2, 3t + t^3)$

$D_1: (-3+1, 3, -3-1) = (-2, 3, -4)$

$D_2: (6 - 8, 12, 6 + 8) = (-2, 12, 14)$

$t_1: \frac{x+2}{0} = \frac{y-3}{-6} = \frac{z+4}{6}$

$t_2: \frac{x+2}{-9} = \frac{y-12}{12} = \frac{z-14}{15}$

6. Pokazati da tangenta krive $\alpha: \mathbb{R} \to E^3$ $\vec{r} = \{t, \frac{t^2}{3}, \frac{2t^3}{27}\}$ zatvara konstantan ugao sa vektorom $\vec{a}\{1, 0, 1\}$; Koliki je taj ugao?

$\vec{r}' = \left(1, \frac{2t}{3}, \frac{\cancel{6}t^2}{\cancel{27}_9}\right) = \left(1, \frac{2}{3}t, \frac{2}{9}t^2\right)$

$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{a} \cdot \vec{r}' = 1 + 0 + \frac{2}{9}t^2 = 1 + \frac{2}{9}t^2$

$|\vec{a}| = \sqrt{1+0+1} = \sqrt{2}$ $\quad |\vec{r}'| = \sqrt{1 + \frac{4}{9}t^2 + \frac{4}{81}t^4} = \sqrt{\left(\frac{2}{9}t^2 + 1\right)^2} = \frac{2}{9}t^2 + 1$

$\cos\varphi = \frac{\cancel{1 + \frac{2}{9}t^2}}{\sqrt{2} \cdot \cancel{\left(\frac{2}{9}t^2 + 1\right)}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \varphi = \frac{\pi}{4}$

7. U kojim tačkama je tangenta krive $\alpha: R \to E^3$ $\vec{r} = \{3t - t^3, 3t^2, 3t + t^3\}$ paralelna s koordinatnim ravnima.

Za xOy: $z = 0$

$\vec{r}' = \{3 - 3t^2, 6t, 3 + 3t^2\}$

$\vec{N} = (0, 0, 1)$

$\vec{r}' \cdot \vec{N} = 0$

$3 + 3t^2 = 0 \Rightarrow 3t^2 = -3 \quad t^2 = -1 \Rightarrow$ ne postoji takva tačka

Za xOz: $y = 0$

$\vec{N} = (0, 1, 0)$

$6t = 0 \Rightarrow t = 0 \qquad M_1(0, 0, 0)$

Za yOz: $x = 0$

$3 - 3t^2 = 0 \Rightarrow 3t^2 = 3 \quad t^2 = 1 \quad t = \pm 1$

$M_2(2, 3, 4) \qquad M_3(-2, 3, -4)$

8. Napisati jed. tangente krive $\alpha: R \to E^3$, $\vec{r} = \{a\cos t, -a\sin t, be^t\}$ i naći geom. mjesto sjecišta tangenti i ravni xOy.

$\vec{r}' = \{-a\sin t, -a\cos t, be^t\}$

$$\frac{x - a\cos t}{-a\sin t} = \frac{y + a\sin t}{-a\cos t} = \frac{z - be^t}{be^t} = \lambda$$

$x = a\cos t - \lambda a \sin t$

$y = -a \sin t - \lambda a \cos t$

$z = b e^{t} + \lambda b e^{t}$

$z = 0$

$b e^{t} + \lambda b e^{t} = 0$

$b e^{t}(1+\lambda) = 0 \Rightarrow (1+\lambda) = 0 \Rightarrow \lambda = -1$

$x = a\cos t + a \sin t = a(\cos t + \sin t)$

$y = -a \sin t + a\cos t = a(\cos t - \sin t)$

$x^2 + y^2 = a^2\cos^2 t + a^2 \sin^2 t + a^2 \sin^2 t + a^2 \cos^2 t =$

$= 2a^2 \underbrace{(\cos^2 t + \sin^2 t)}_{1} = 2a^2 \rightarrow$ kružnica

1. Naći oskulacionu ravan krive $y^2 = x,\ x^2 = z$ u tački $T(1,1,1)$.

$y = t \qquad \dot{y} = 1 \qquad \ddot{y} = 0$

$x = t^2 \qquad \dot{x} = 2t \qquad \ddot{x} = 2$

$z = t^4 \qquad \dot{z} = 4t^3 \qquad \ddot{z} = 12t^2$

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \\ \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - t^2 & y - t & z - t^4 \\ 2t & 1 & 4t^3 \\ 2 & 0 & 12t^2 \end{vmatrix} = -(y-t)\underbrace{(24t^3 - 8t^3)}_{16t^3} + (x - t^2)12t^2 - 2(z - t^4) =$$

$$= -16t^3 y + 16t^4 + 12t^2 x - 12t^4 - 2z + 2t^4 =$$

$$= 6t^4 - 16t^3 y + 12t^2 x - 2z$$

$$6t^4 - 16 y t^3 + 12 x t^2 - 2z = 0$$

$6t^4 - 16t^3 + 12t^2 - 2 = 0 \quad /:2$

$3t^4 - 8t^3 + 6t^2 - 1 = 0$

	3	-8	6	0	-1
1	3	-5	1	1	0

$t_1 = 1$

$3t^3 - 5t^2 + t + 1 = 0$

	3	-5	1	1
1	3	-2	-1	0

$t_2 = 1$

$3t^2 - 2t - 1 = 0$

$t_{1,2} = \frac{2 \pm \sqrt{4+12}}{6} = \frac{2 \pm 4}{6} \begin{cases} 1 \\ -\frac{1}{3} \end{cases}$

$t_3 = 1 \quad t_4 = -\frac{1}{3}$

$t = 1$

$12x - 16y - 2z + 6 = 0 \quad /:2$

$6x - 8y - z + 3 = 0$

$t = -\frac{1}{3}$

$12 \cdot \frac{1}{9}x - 16 \cdot \frac{-1}{27}y - 2z + 6 \cdot \frac{1}{81} = 0$

$\frac{4}{3}x + \frac{16}{27}y - 2z + \frac{2}{27} = 0$

ovo bi radili da nam nije data

$T(0, 1, 1)$

2. Naći normalnu ravan krive $z=x^2+y^2$, $y=x$, u tački $x=1$

3. Naći jed. binormale krive $\alpha: \mathbb{R} \to E^3$ zadana kao $x=t-\sin t$, $y=1-\cos t$, $z=4\cos\frac{t}{2}$ u tački $t=\pi$

Rj. 2. $x+y+4z-10=0$; 3. $\frac{x-\pi}{1}=\frac{y-2}{0}=\frac{z-0}{1}$

4. Napisati jed. tangente, glavne normale i binormale za krivu $\alpha: \mathbb{R} \to E^3$ zadanu: $x=a\cos t$, $y=a\sin t$, $z=bt$ u proizvoljnoj tački.
Dokazati da glavna normala sječe ovu krivu pod pravim uglom, a binormala zatvara konstantan ugao.

jed. tang. $\frac{x-a\cos t}{-a\sin t}=\frac{y-a\sin t}{a\cos t}=\frac{z-bt}{b}$

glavna norm.: $\frac{x-a\cos t}{\cos t}=\frac{y-a\sin t}{\sin t}=\frac{z-bt}{0}$

binormala $\frac{x-a\cos t}{b\sin t}=\frac{y-a\sin t}{-b\cos t}=\frac{z-bt}{a}$

$\cos\varphi=0 \Rightarrow \varphi=\frac{\pi}{2}$

xOy $\vec{N}(0,0,1)$
$\vec{b}$

$\boxed{\cos\varphi=\frac{a}{\sqrt{a^2+b^2}}}$

$\vec{N}\cdot\vec{b}=a$
$|\vec{b}|=\sqrt{b^2+a^2}$ $\quad |\vec{N}|=1$ $\quad \cos\varphi=\frac{a}{\sqrt{b^2+a^2}}$

5. Napisati jed. oskulacione ravni krive $\alpha: \mathbb{R} \to E^3$ zadane vektorski: $(e^t, e^{-t}, t\sqrt{2})$ Rj. $e^{-t}x-e^t y-\sqrt{2}z+2t=0$

6. Napisati jed. oskulacione ravni krive $x^2+y^2+z^2=9$ i $x^2-y^2=3$ u tački $M(2,1,2)$. kao h. iz tangente

$2x+2y\cdot\dot{y}+2z\cdot\dot{z}=0$ $\qquad 4+2\cdot 2+4\dot{z}=0$ $\qquad \vec{r}=\{x, y(x), z(x)\}$

$2x-2y\cdot\dot{y}=0$ $\qquad 4\dot{z}=-8$ $\qquad \dot{x}=1$

$4+2\dot{y}+4\dot{z}=0$ $\qquad \boxed{\dot{z}=-2}$

$4-2\dot{y}=0 \Rightarrow \boxed{\dot{y}=2}$

$2+2\dot{y}\cdot\dot{y}+2y\cdot\ddot{y}+2\dot{z}\cdot\dot{z}+2z\cdot\ddot{z}=0$ $\quad \dot{x}=1 \quad \dot{y}=2 \quad \dot{z}=-2$

$\cancel{2=2\dot{y}\dot{y}+y\ddot{y}=0}$ $\quad x=2 \quad y=1 \quad z=2$

$2+8+2\ddot{y}+8+4\ddot{z}=0$ $\quad \ddot{x}=0$

$2-8-2\ddot{y}=0 \Rightarrow -2\ddot{y}=6 \Rightarrow \boxed{\ddot{y}=-3}$

$18-6+4\ddot{z}=0 \quad 4\ddot{z}=-12 \Rightarrow \boxed{\ddot{z}=-3}$

$$\begin{vmatrix} x-2 & y-1 & z-2 \\ 1 & 2 & -2 \\ 0 & -3 & -3 \end{vmatrix} = (x-2)(-6-6)+\left[(y-1)\cdot(-3)-(-3)(z-2)\right]=$$

$$=-12x+24-\left[-3y+3+3z-6\right]=$$

$$=-12x+24+3y-3-3z+6=$$

$$=-12x+3y-3z+27$$

$4x-y+z-9=0$

7. Napisati jed. rektifikacione ravni $\alpha:(0,\pi)\to E^3$

$\vec{r}=\left\{\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}},\ln \sin t\right\} \quad t=\frac{\pi}{2}$.

$l=\begin{vmatrix} \dot{y}_0 & \dot{z}_0 \\ \ddot{y}_0 & \ddot{z}_0 \end{vmatrix} \quad m=\begin{vmatrix} \dot{z}_0 & \dot{x}_0 \\ \ddot{z}_0 & \ddot{x}_0 \end{vmatrix} \quad n=\begin{vmatrix} \dot{x}_0 & \dot{y}_0 \\ \ddot{x}_0 & \ddot{y}_0 \end{vmatrix}$

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \\ l & m & n \end{vmatrix}=0 \qquad \vec{r}=\left\{\frac{\pi}{2\sqrt{2}},\frac{\pi}{2\sqrt{2}},0\right\}$$

$\dot{x}=\frac{1}{\sqrt{2}} \quad \dot{y}=\frac{1}{\sqrt{2}} \quad \dot{z}=\frac{\cos t}{\sin t}=\text{ctg}\, t$

$\ddot{x}=0 \quad \ddot{y}=0 \quad \ddot{z}=-\frac{1}{\sin^2 t}$

$l=\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{\cos t}{\sin t} \\ 0 & -\frac{1}{\sin^2 t} \end{vmatrix}=-\frac{1}{\sqrt{2}\sin^2 t} \qquad n=\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{vmatrix}=0$

$m=\begin{vmatrix} \frac{\cos t}{\sin t} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sin^2 t} & 0 \end{vmatrix}=\frac{1}{\sqrt{2}\sin^2 t}$

$$\begin{vmatrix} x-\frac{t}{\sqrt2} & y-\frac{t}{\sqrt2} & z-\ln\sin t \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & \frac{\cos t}{\sin t} \\ -\frac{1}{\sqrt2\sin^2 t} & -\frac{1}{\sqrt2\sin^2 t} & +0 \end{vmatrix} = (z-\ln\sin t)\cdot\left[\frac{1}{2\sin^2 t}+\frac{1}{2\sin^2 t}\right] - \frac{\cos t}{\sin t}\cdot\left[\frac{x}{\sqrt2\sin^2 t}-\frac{t}{2\sin^2 t}+\frac{y}{\sqrt2\sin^2 t}-\frac{t}{2\sin^2 t}\right]$$

$= z-\underbrace{\ln 1}_{0}\cdot 1 - 0 \to \boxed{z=0} \to$ rektif. ravan

$$\cancel{\operatorname{ctg}t\left(x-\frac{t}{\sqrt2}\right)+\operatorname{ctg}t\left(y-\frac{t}{\sqrt2}\right)-\sqrt2(z-\ln\sin t)=0}$$

FLEKSIJA I TORZIJA

Fleksiju zovemo i prvom zakrivljenosti i geometrijski ona predstavlja mjeru za zaokret tang. vektora. Torzija (II zakrivljenost) geometrijski predstavlja mjeru zaokreta oskulacione ravni kad se tačka dodira pomjera duž krive.

1. Naći poluprečnik zakrivljenosti krive:

$x=a(t-\sin t)$ $\quad y=a(1-\cos t)$ $\quad z=4a\sin\frac{t}{2}$

$\dot x=a(1-\cos t)$ $\quad \dot y=a\sin t$ $\quad \dot z=4a\cdot\frac{1}{2}\cos\frac{t}{2}=2a\cos\frac{t}{2}$

$\ddot x=a\sin t$ $\quad \ddot y=a\cos t$ $\quad \ddot z=-a\sin\frac{t}{2}$

$$\dot x^2+\dot y^2+\dot z^2 = a^2(1-2\cos t+\cos^2 t)+a^2\sin^2 t+4a^2\cos^2\frac{t}{2}=$$

$$=a^2-2a^2\cos t+\underbrace{a^2\cos^2 t+a^2\sin^2 t}_{a^2}+4a^2\cos^2\frac{t}{2}=$$

$$=2a^2\left(1-\cos t+2\cos^2\frac{t}{2}\right)=2a^2\left(2\sin^2\frac{t}{2}+2\cos^2\frac{t}{2}\right)=$$

$$=4a^2$$

$$\ddot{x}^2+\ddot{y}^2+\ddot{z}^2 = a^2\left(\underbrace{\sin^2 t+\cos^2 t}_{1}+\sin^2\frac{t}{2}\right) = a^2\left(1+\sin^2\frac{t}{2}\right)$$

$$\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z} = (a-a\cos t)\,a\sin t + a^2\sin t\cos t - 2a^2\sin\frac{t}{2}\cos\frac{t}{2}$$

$$= a^2\sin t - \cancel{a^2\sin t\cos t} + \cancel{a^2\sin t\cos t} - 2a^2\sin\frac{t}{2}\cos\frac{t}{2}$$

$$= a^2\sin t - a^2\sin t = 0$$

$$\chi(t) = \sqrt{\frac{4a^2\cdot a^2\left(1+\sin^2\frac{t}{2}\right)-0}{(4a^2)^3}} = \sqrt{\frac{4a^4\left(1+\sin^2\frac{t}{2}\right)}{4^{2}\cancel{4}a^{6}}} = \frac{\sqrt{1+\sin^2\frac{t}{2}}}{4a} =$$

$$\frac{1}{\chi(t)} = \frac{4a}{\sqrt{1+\sin^2\frac{t}{2}}}$$

2. Pokazati da su zakrivljenost i torzija krive

$x = a\cos t$, $y = a\sin t$, $z = bt$, $t\in R$ (a,b) constantne

$\dot{x} = -a\sin t$ $\quad\dot{y} = a\cos t$ $\quad\dot{z} = b$

$\ddot{x} = -a\cos t$ $\quad\ddot{y} = -a\sin t$ $\quad\ddot{z} = 0$

$\dddot{x} = a\sin t$ $\quad\dddot{y} = -a\cos t$ $\quad\dddot{z} = 0$

$$\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = b\left(a^2\cos^2 t + a^2\sin^2 t\right) = ba^2 = a^2 b$$

$$(\dot{x}^2+\dot{y}^2+\dot{z}^2) = a^2\sin^2 t + a^2\cos^2 t + b^2 = a^2+b^2$$

$$(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2) = a^2\cos^2 t + a^2\sin^2 t + 0 = a^2$$

$$\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z} = \cancel{a^2\sin t\cos t} - \cancel{a^2\sin t\cos t} + b\cdot 0 = 0$$

$$\tau(t) = \frac{\cancel{a^2}b}{(a^2+b^2)\cancel{a^2}} = \frac{b}{a^2+b^2}$$

$$\chi(t) = \sqrt{\frac{(a^2+b^2)a^2}{(a^2+b^2)^{3}}} = \frac{a}{a^2+b^2}$$

3. Naći zakrivljenost i torziju krive $x=2t$, $y=\ln t$, $z=t^2$, $t\in\mathbb{R}$ u tački $(2,0,1)$

$\dot{x}=2$ $\quad \dot{y}=\frac{1}{t}$ $\quad \dot{z}=2t$

$\ddot{x}=0$ $\quad \ddot{y}=\frac{-1}{t^2}$ $\quad \ddot{z}=2$

$\dddot{x}=0$ $\quad \dddot{y}=\frac{2}{t^3}$ $\quad \dddot{z}=0$

$2=2t$ $\quad 0=\ln t$ $\quad 1=t^2$

$t=1$ $\quad t=1$ $\quad t=\pm 1$

$\dot{x}^2+\dot{y}^2+\dot{z}^2=4+1+4=9$

$\ddot{x}^2+\ddot{y}^2+\ddot{z}^2=0+1+4=5$

$\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}=0-1+4=3$

$$\mathcal{K}(t)=\sqrt{\frac{9\cdot 5-3^2}{9^3}}=\sqrt{\frac{45-9}{9^3}}=\frac{6}{9\cdot\cancel{3}}=\frac{2}{9}$$

$$\begin{vmatrix} 2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 2 & 0 \end{vmatrix}=2(0-4)=-8$$

$$\tau(t)=\frac{-8}{9\cdot 5-9}=\frac{-8}{36}=\frac{-2}{9}$$

4. Naći zakrivljenost i torziju krive $x^3=3a^2y$, $2xz=a^2$

$$x=t \qquad y=\frac{t^3}{3a^2} \qquad z=\frac{a^2}{2t}$$

$$\dot{x}=1 \qquad \dot{y}=\frac{\cancel{3}t^2}{\cancel{3}a^2}=\frac{t^2}{a^2} \qquad \dot{z}=-\frac{a^2}{2t^2}$$

$$\ddot{x}=0 \qquad \ddot{y}=\frac{2t}{a^2} \qquad \ddot{z}=\frac{2a^2}{2t^3}=\frac{a^2}{t^3}$$

$$\dddot{x}=0 \qquad \dddot{y}=\frac{2}{a^2} \qquad \dddot{z}=\frac{-3a^2}{t^4}$$

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=1+\frac{t^4}{a^4}+\frac{a^4}{4t^4}=\frac{4a^4t^4+4t^8+a^8}{4a^4t^4}=\frac{(2t^4+a^4)^2}{4a^4t^4}$$

$$\ddot{x}^2+\ddot{y}^2+\ddot{z}^2=0+\frac{4t^2}{a^4}+\frac{a^4}{t^6}=\frac{4t^8+a^8}{a^4t^6}$$

$$(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})^2=\left(0+\frac{2t^3}{a^4}-\frac{a^4}{2t^5}\right)^2=\left(\frac{4t^8-a^8}{2a^4t^5}\right)^2$$

$$\begin{vmatrix} 1 & \frac{t^2}{a^2} & -\frac{a^2}{2t^2} \\ 0 & \frac{2t}{a^2} & \frac{a^2}{t^3} \\ 0 & \frac{2}{a^2} & \frac{-3a^2}{t^4} \end{vmatrix} = \frac{-6\cancel{a^2}\cancel{t}}{\cancel{a^2}t^{\cancel{4}\,3}}-\frac{2\cancel{a^2}}{\cancel{a^2}t^3}=-\frac{6}{t^3}-\frac{2}{t^3}=\frac{-8}{t^3}$$

$$\mathcal{X}(t)=\sqrt{\frac{\frac{(2t^4+a^4)^2}{4a^4t^4}\cdot\frac{4t^8+a^8}{a^4t^6}-\left(\frac{4t^8-a^8}{2a^4t^5}\right)^2}{\left(\frac{(2t^4+a^4)^2}{4a^4t^4}\right)^3}}$$

$$\frac{4a^4t^4+4t^8+a^8}{4a^4t^4}\cdot\frac{4t^8+a^8}{a^4t^6}=\frac{16a^4t^{12}+16t^{16}+4a^8t^8+4a^{12}t^4+4a^8t^8+a^{16}}{4a^8t^{10}}$$

$$\frac{16t^{16}-8a^8t^8+a^{16}}{4a^8t^{10}}$$

Moje Rj.: $\chi(t) = \dfrac{8t^3}{a(2t^4+1)^2}$ $\tau(t) = -\dfrac{8t^3}{a(2t^4+1)}$

5. Naći poluprečnik zakrivljenosti i torziju krive u tački $(1,1,1)$; $\ldots y^2 + z^2 = 1$ $\quad y^2 - 2x + z = 0$

$x = 0$

$2x - 2y \cdot \dot y + 2z \cdot \dot z = 0$

$2y \cdot \dot y - 2 + z' = 0$

$2y \cdot \dot y - 2 + z = 0$

$x = 1 \quad y = 1 \quad z = 1$

$2 - 2\dot y + 2\dot z = 0 \quad / : 2$

$2\dot y - 2 + \dot z = 0$

$\dot z = \dot y - 1$

$2\dot y - 2 + \dot y - 1 = 0$

$3\dot y = 3 \quad \dot y = 1 \quad \dot z = 0 \quad \dot x = 1 \qquad M'(1,1,0)$

$2 - (2\dot y \dot y + 2y \cdot \ddot y) + 2\dot z \cdot \dot z + 2z \cdot \ddot z = 0$

$2\dot y \dot y + 2y \cdot \ddot y + \ddot z = 0$

$\cancel{2} - \cancel{2} - 2\ddot y + 2\ddot z = 0$

$2 + 2\ddot y + \ddot z = 0$

$2 + 3\ddot z = 0 \Rightarrow \ddot z = \frac{-2}{3}$

$2 + 2\ddot y - \frac{2}{3} = 0 \Rightarrow 2\ddot y = \frac{2}{3} - 2 \qquad \ddot y = \frac{-4}{3 \cdot 2} = \frac{-2}{3}$

$\ddot x = 0$

$\varkappa(t) = \frac{1}{\sqrt{6}}$

$\frac{1}{\varkappa(t)} = \sqrt{6} \qquad \tau(t) = 1$

1. Naći zakrivljenost i torziju krive: $y=\frac{x^2}{2a}$, $z=\frac{x^3}{6a^2}$ u točki $x=2a$.

$$x=t \qquad y=\frac{t^2}{2a} \qquad z=\frac{t^3}{6a^2}$$

$$\dot{x}=1 \qquad \dot{y}=\frac{2t}{2a}=\frac{t}{a} \qquad \dot{z}=\frac{\cancel{3}t^2}{\cancel{6}_2a^2}=\frac{t^2}{2a^2}$$

$$\ddot{x}=0 \qquad \ddot{y}=\frac{1}{a} \qquad \ddot{z}=\frac{t}{a^2}$$

$$\dddot{x}=0 \qquad \dddot{y}=0 \qquad \dddot{z}=\frac{1}{a^2}$$

$$\varkappa(t)=\sqrt{\frac{\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)\left(0+\frac{1}{a^2}+\frac{t^2}{a^4}\right)-\left(0+\frac{t}{a^2}+\frac{t^3}{2a^4}\right)^2}{\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)^3}}$$

$$\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)\left(\frac{1}{a^2}+\frac{t^2}{a^4}\right)=\frac{1}{a^2}+\frac{t^2}{a^4}+\frac{t^2}{a^4}+\frac{t^4}{a^6}+\frac{t^4}{4a^6}+\frac{t^6}{4a^8}=$$

$$=\frac{1}{a^2}+\frac{2t^2}{a^4}+\frac{5t^4}{4a^6}+\frac{t^6}{4a^8}$$

$$\left(\frac{t}{a^2}+\frac{t^3}{2a^4}\right)^2=\frac{t^2}{a^4}+\cancel{2}\cdot\frac{t^4}{\cancel{2}a^6}+\frac{t^6}{4a^8}$$

$$\frac{1}{a^2}+\frac{2t^2}{a^4}+\frac{5t^4}{4a^6}+\cancel{\frac{t^6}{4a^8}}-\frac{t^2}{a^4}-\frac{t^4}{a^6}-\cancel{\frac{t^6}{4a^8}}=\frac{1}{a^2}+\frac{t^2}{a^4}+\frac{5t^4-4t^4}{4a^6}=$$

$$=\frac{1}{a^2}+\frac{t^2}{a^4}+\frac{t^4}{4a^6}$$

$$\varkappa(t)=\sqrt{\frac{\frac{1}{a^2}\cancel{\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)}}{\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)^{\cancel{3}\,2}}}=\frac{1}{a\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)}$$

$$\varkappa(x=2a)=\frac{1}{a\left(1+\frac{4a^2}{a^2}+\frac{16a^4}{4a^4}\right)}=\frac{1}{a(1+4+4)}=\frac{1}{9a}$$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} 1 & \frac{t}{a} & \frac{t^2}{2a^2} \\ 0 & \frac{1}{a} & \frac{t}{a^2} \\ 0 & 0 & \frac{1}{a^2} \end{vmatrix} = \frac{1}{a}\cdot\frac{1}{a^2} = \frac{1}{a^3}$$

$$\tau = \frac{\frac{1}{a^3}}{\frac{1}{a^2}\left(1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}\right)} = \frac{\frac{1}{a}}{1+\frac{t^2}{a^2}+\frac{t^4}{4a^4}}$$

$$\tau(x=2a=t) = \frac{\frac{1}{a}}{1+\frac{4a^2}{a^2}+\frac{16a^4}{4a^4}} = \frac{1}{9a}$$

2. Naći zakrivljenost i poluprečnik zakrivljenosti krive $x^2+y^2+z^2-4=0$, $x+y-z=0$ u tački:

$x=0,\ y>0,\ z>0$.

$x=0 \Rightarrow \left.\begin{array}{l} y^2+z^2-4=0 \\ y-z=0 \end{array}\right\}$ - sis.; rj. su >0

kada dobijemo tačku vršimo param. $x=t$

Rj.: $\mathcal{K}(t)=\frac{1}{2}$; $\frac{1}{\mathcal{K}(t)}=2$

3. Naći tačke na krivoj $x=\cos^3 t$, $y=\sin^3 t$, $z=\cos 2t$, $t\in[0,\pi]$ u kojima zakrivljenost prima minimalnu vrijednost

Prvo nađemo fleksiju

Rj.: $\mathcal{K}(t)=\frac{3}{25}\cdot\frac{1}{\sin t\cdot\cos t}$

$\mathcal{K}(t)$ prima ~~min.~~ vrijednost kada je $\sin t\cdot\cos t$ maximalno tj. u $\frac{\pi}{4}$

VEKTORSKIM FORMULAMA

4. Pokazati da se normalna, rektifikaciona i oskulaciona ravan krive $x=3t$, $y=3t^2$, $z=2t^3$, $t\in R$, u tački maximalne zakrivljenosti podudaraju sa koordinatnim ravnima.

Rj. $\mathcal{X}(t)=\frac{2}{3\cdot(1+2t^2)^2}$ za $t=0 \Rightarrow$ max fleksija

pa tražimo ravni

5. Zadana je kriva $x=a\cos t$, $y=a\sin t$, ~~$z=\frac{a}{\sqrt2}$~~

$z=\frac{a}{\sqrt2}(\sin t+\cos t)$, $t\in R$, u tački $N(t)$ naći oskulacionu ravan, glavnu normalu, polupr. zakrivljenosti i torziju.

Rj.: $\frac{1}{\mathcal{X}(t)}=\frac{a^3\sqrt{\left(\frac{3}{2}-\sin t\cos t\right)^3}}{\sqrt2}$; $\tau=0$

6. Naći jed. oskulacione ravni, glavne normale, polupr. zakrivljenosti i torziju krive $x^2=y$, $x^3=z$ u jednoj njezinoj tački.

Oskulaciona ravan: $3t^2x-3ty+z-t^3=0$

Glavna normala: $\frac{x-t}{9t^3+2t}=\frac{y-t^2}{9t^4-1}=\frac{z-t^3}{-6t^3-3t}$

Poluprečnik: $\frac{1}{\mathcal{X}(t)}=\frac{(1+4t^2+9t^4)^{3/2}}{2(1+9t^2+9t^4)^{1/2}}$

Torzija: $\tau=\frac{3}{4(9t^4+9t^2+1)}$

5. $x = a\cos t$, $y = a\sin t$, $z = \frac{a}{\sqrt{2}}(\sin t + \cos t)$

$\dot{x} = -a\sin t$ $\quad \dot{y} = a\cos t$ $\quad \dot{z} = \frac{a}{\sqrt{2}}(\cos t - \sin t)$

$\ddot{x} = -a\cos t$ $\quad \ddot{y} = -a\sin t$ $\quad \ddot{z} = \frac{a}{\sqrt{2}}(-\sin t - \cos t)$

$\dddot{x} = a\sin t$ $\quad \dddot{y} = -a\cos t$ $\quad \dddot{z} = \frac{a}{\sqrt{2}}(-\cos t + \sin t)$

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \\ \ddot{x}_0 & \ddot{y}_0 & \ddot{z}_0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - a\cos t & y - a\sin t & z - \frac{a}{\sqrt{2}}(\sin t + \cos t) \\ -a\sin t & a\cos t & \frac{a}{\sqrt{2}}(\cos t - \sin t) \\ -a\cos t & -a\sin t & \frac{a}{\sqrt{2}}(-\sin t - \cos t) \end{vmatrix} =$$

$$= (x - a\cos t)\left[\frac{a^2\cos t}{\sqrt{2}}(-\sin t - \cos t) - \frac{a^2\sin t}{\sqrt{2}}(\cos t - \sin t)\right]$$

$$-(-a\sin t)$$

$$\vec{r}_0 = \left(a\cos t,\ a\sin t,\ \frac{a}{\sqrt{2}}(\sin t + \cos t)\right)$$

$$\dot{\vec{r}}_0 = \left(-a\sin t,\ a\cos t,\ \frac{a}{\sqrt{2}}(\cos t - \sin t)\right)$$

$$\ddot{\vec{r}}_0 = \left(-a\cos t,\ -a\sin t,\ \frac{a}{\sqrt{2}}(-\sin t - \cos t)\right)$$

$$(\vec{r} - \vec{r}_0)\cdot(\dot{\vec{r}}_0 \times \ddot{\vec{r}}_0) = 0$$

$$\dot{\vec{r}}_0 \times \ddot{\vec{r}}_0 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & \frac{a}{\sqrt{2}}(\cos t - \sin t) \\ -a\cos t & -a\sin t & \frac{a}{\sqrt{2}}(-\sin t - \cos t) \end{vmatrix} =$$

$$= \vec{i}\left(\frac{a^2\cos t}{\sqrt{2}}(-\sin t - \cos t) + \frac{a^2\sin t}{\sqrt{2}}(\cos t - \sin t)\right) -$$

$$\vec{j}\left(\frac{-a^2\sin t}{\sqrt{2}}(-\sin t - \cos t) + \frac{a^2\cos t}{\sqrt{2}}(\cos t - \sin t)\right) +$$

$$\vec{k}\left(a^2\underbrace{(\sin^2 t + \cos^2 t)}_{1}\right)$$

$$= \vec{i}\cdot\frac{a^2}{\sqrt{2}}\left(\cancel{-\sin t\cos t} - \cos^2 t + \cancel{\sin t\cos t} - \sin^2 t\right) -$$

$$\vec{j}\cdot\frac{a^2}{\sqrt{2}}\left(\sin^2 t + \cancel{\sin t\cos t} + \cos^2 t - \cancel{\sin t\cos t}\right) + a^2\vec{k}$$

$$= -\frac{a^2}{\sqrt{2}}\vec{i} - \frac{a^2}{\sqrt{2}}\vec{j} + a^2\vec{k}$$

$$\left(x - a\cos t,\ y - a\sin t,\ z - \frac{a}{\sqrt{2}}(\sin t + \cos t)\right)\cdot\left(\frac{-a^2}{\sqrt{2}}, \frac{-a^2}{\sqrt{2}}, a^2\right) = 0$$

$$-\frac{a^2 x}{\sqrt{2}} + \cancel{\frac{a^3\cos t}{\sqrt{2}}} - \frac{a^2 y}{\sqrt{2}} + \cancel{\frac{a^3\sin t}{\sqrt{2}}} + a^2 z - \cancel{\frac{a^3\sin t}{\sqrt{2}}} - \cancel{\frac{a^3\cos t}{\sqrt{2}}} = 0 \ /\cdot\sqrt{2}$$

$$-a^2 x - a^2 y + \sqrt{2}a^2 z = 0 \quad /\cdot\frac{-1}{a^2}$$

$x + y - \sqrt{2}z = 0 \rightarrow$ oskulaciona ravan

glavna normala: $\dfrac{x - a\cos t}{-a\cos t} = \dfrac{y - a\sin t}{-a\sin t} = \dfrac{z - \frac{a}{\sqrt{2}}(\sin t + \cos t)}{\frac{a}{\sqrt{2}}(\sin t - \cos t)}$

$$|\dot{\vec{r}}_0 \times \ddot{\vec{r}}_0| = \sqrt{\frac{a^4}{2} + \frac{a^4}{2} + a^4} = a^2\sqrt{2}$$

$$|\dot{\vec{r}}| = \sqrt{a^2 + \frac{a^2}{2}(\cos^2 t - 2\sin t\cos t + \sin^2 t)} = a\sqrt{1 + \frac{1}{2}(1 - 2\sin t\cos t)}$$

$$x(t) = \frac{\cancel{a^2}\sqrt{2}}{\cancel{a^3}\sqrt{\left(1+\frac{1}{2}-\sin t\cos t\right)^3}} = \frac{\sqrt{2}}{a\sqrt{\left(\frac{3}{2}-\sin t\cos t\right)^3}}$$

$$\frac{1}{x(t)} = \frac{a\sqrt{\left(\frac{3}{2}-\sin t\cos t\right)^3}}{\sqrt{2}}$$

$$\tau(t) = \frac{(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}})}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$\begin{vmatrix} -a\sin t & a\cos t & \frac{a}{\sqrt{2}}(\cos t-\sin t) \\ -a\cos t & -a\sin t & \frac{a}{\sqrt{2}}(-\sin t-\cos t) \\ a\sin t & -a\cos t & \frac{a}{\sqrt{2}}(-\cos t+\sin t) \end{vmatrix} =$$

$$= -a\sin t\left(\frac{-a^2(\cancel{-\sin t\cos t}+\sin^2 t+\cancel{\sin t\cos t}+\cos^2 t)}{\sqrt{2}}\right)$$

$$-a\cos t\left(\frac{+a^2(-\cos^2 t+\cancel{\sin t\cos t}-\sin^2 t-\cancel{\sin t\cos t})}{\sqrt{2}}\right)$$

$$+\frac{a}{\sqrt{2}}(\cos t-\sin t)\cdot(a^2\cos^2 t+a^2\sin^2 t)$$

$$= +\frac{a\sin t\cdot a^2}{\sqrt{2}} - \frac{a\cos t\cdot a^2}{\sqrt{2}} + \frac{a}{\sqrt{2}}\cdot a^2\cdot(\cos t-\sin t)$$

$$= \frac{a^3}{\sqrt{2}}(\sin t-\cos t+\cos t-\sin t) = 0$$

$$\tau(t) = 0$$

PLOHE

1. Zadana je sfera implicitnom jed.:
$(x-c_1)^2+(y-c_2)^2+(z-c_3)^2=r^2$ gdje je $C(c_1,c_2,c_3)$ središte sfere a r poluprečnik. Dokazati da je sfera ploha.

$F \equiv (x-c_1)^2+(y-c_2)^2+(z-c_3)^2=r^2$

$dF \neq 0$

$dF = 2(x-c_1)+2(y-c_2)+2(z-c_3)=0$

$x=c_1, \; y=c_2$ i $z=c_3$

Ove jednakosti su nemoguće jer u uslovu zad. je rečeno da je tačka $C(c_1,c_2,c_3)$ središte sfere a mi ovim jednakostima dobijemo da bi ovo bila tačka na sferi, što je kontradiktorno $\Rightarrow dF \neq 0 \Rightarrow$ sfera je ploha.

2. Data je ravan $3x+4y-2z-3=0$. Naći njene parametarske jednačine i to tako da su kolinearne krive dvije familije međusobno okomitih pravaca

$\vec{u}\cdot\vec{v}=0, \quad \vec{N}=(3,4,-2)$

Uzmimo proizvoljnu tačku u ravni:

$T(1,2,z)$

$-2z=3-4y-3x=3-8-3 \Rightarrow z=4 \Rightarrow T(1,2,4)$

1° Prvu koordinatnu osu u odabrat ćemo tako da leži u ravni tj. da prolazi tačkom T i da je ispunjen uslov $\vec{N}\cdot\vec{u}=0$

$\vec{u}=(l_1, m_1, n_1)$

$\vec{v}=(l_2, m_2, n_2)$

Proizvoljno $\vec{u}\ (6, -2, n_1)$

$(3, 4, -2)\cdot(6, -2, n_1)=0$

$18-8-2n_1=0 \quad n_1=5 \quad \Rightarrow \quad \vec{u}=(6, -2, 5)$

Prva koordinatna kriva je:

$(u): \ \frac{x-1}{6}=\frac{y-2}{-2}=\frac{z-4}{5}$

2. koord. krivu ćemo dobiti iz 2 uslova:

$\vec{u}\cdot\vec{v}=0 \rightarrow$ dat u zad.

$\vec{N}\cdot\vec{v}=0 \leftarrow$ uzimamo kao i za u:

$(6, -2, 5)\cdot(l_2, m_2, n_2)=0$

$(3, 4, -2)\cdot(l_2, m_2, n_2)=0$

$6l_2-2m_2+5n_2=0$

$3l_2+4m_2-2n_2=0$

$18l_2=-8n_2$

$l_2=-\frac{8}{18}n_2$

$6l_2-2m_2=-5n_2 \quad /:n_2\neq 0$

$3l_2+4m_2=2n_2 \quad /:n_2\neq 0$

$6\frac{l_2}{n_2}-2\frac{m_2}{n_2}=-5 \quad /\cdot 2$

$3\frac{l_2}{n_2}+4\frac{m_2}{n_2}=2$

$6t+2s=-5 \quad /\cdot 2$

$3t+4s=2$

$15t=-8 \quad t=$

$15\frac{l_2}{n_2}=-8$

$\frac{l_2}{n_2}=-\frac{8}{15}$

$l_2=-\frac{8}{15}n_2$

$3\cdot\frac{-8}{15}+4\cdot\frac{m_2}{n_2}=2$

$\frac{m_2}{n_2}=\frac{2+\frac{8}{5}}{4}=\frac{\frac{18}{5}}{4}=\frac{9}{10}$

$m_2=\frac{9}{10}n_2$

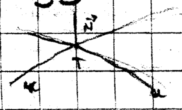
$$(v): \frac{x-1}{-\frac{8}{15}m_2} = \frac{y-2}{\frac{9}{10}m_2} = \frac{z-4}{m_2}$$

$$x = 1 + 6u - 16v$$

$$(v): \frac{x-1}{-16} = \frac{y-2}{27} = \frac{z-4}{30}$$

$$y = 2 - 2u + 27v$$

$$z = 4 + 5u + 30v$$



3. Zadana je ploha:

$$\vec{r} = \{e^{av} f(u)\cos(u+v),\ e^{av} f(u)\sin(u+v),\ e^{av} g(u)\}$$

$u, v \in \mathbb{R}$; $f(u), g(u)$ su proizvoljne fje. Pokazati da koord. krive za $u = c$ leže na krivoj oblika:

$$x^2 + y^2 = \frac{f^2(c)}{g^2(c)} z^2 \quad \text{(kupa ili stožac u prostoru)}$$

$$x^2 + y^2 = e^{2av} f^2(u) \underbrace{\left[\cos^2(u+v) + \sin^2(u+v)\right]}_{1} = e^{2av} f^2(u)$$

$$z^2 = e^{2av} g^2(u) \Rightarrow e^{2av} = \frac{z^2}{g^2(u)}$$

$$x^2 + y^2 = \frac{z^2}{g^2(u)} \cdot f^2(u) \xRightarrow{u=c} x^2 + y^2 = \frac{f^2(c)}{g^2(c)} \cdot z^2$$

4. Naći singularnu krivu na pseudosferi:

$$x = \sin u \cos v$$

$$y = \sin u \sin v$$

$$z = \cos u + \ln \operatorname{tg} \frac{u}{2}$$

gdje je $u \geq 0$, $v \in [0, 2\pi]$

$$\frac{\partial x}{\partial u} = \cos v \cdot \cos u$$

$$\frac{\partial y}{\partial u} = \sin v \cdot \cos u$$

$$\frac{\partial z}{\partial u} = \sin u + \frac{1}{2 \cdot \operatorname{tg}\frac{u}{2}} \cdot \frac{1}{\cos^2\frac{u}{2}} = \sin u + \frac{1}{2 \cdot \frac{\sin\frac{u}{2}}{\cos\frac{u}{2}} \cdot \cos^2\frac{u}{2}} = \sin u + \frac{1}{\sin u}$$

$$\frac{\partial x}{\partial v} = -\sin u \sin v$$

$$\frac{\partial y}{\partial v} = \sin u \cos v$$

$$\frac{\partial z}{\partial v} = 0$$

$$\vec{r}_u \times \vec{r}_v = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{b} \\ \cos u \cos v & \cos u \sin v & -\sin u + \dfrac{1}{\sin u} \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} =$$

$$= \vec{i} \cdot \left(\sin^2 u \cos v - \cos v \right) + \vec{j} \left(\sin^2 u \sin v - \sin v \right) + \vec{b} \left(\sin u \cos u \underbrace{(\sin^2 v + \cos^2 v)}_{1} \right)$$

$$\sin^2 u \cos v - \cos v = 0$$

$$\sin^2 u \sin v - \sin v = 0$$

$$\sin u \cos u = 0$$

$$\cos v \left(\sin^2 u - 1 \right) = 0$$

$$\sin v \left(\sin^2 u - 1 \right) = 0$$

$$\sin u \cos u = 0$$

$$\sin u \cos v - \sin u \cos v \cdot \frac{1}{2 \operatorname{tg} \frac{u}{2} \cos^2 \frac{u}{2}} = 0$$

$$\sin^2 u \sin v - \frac{\sin u \sin v}{2 \operatorname{tg} \frac{u}{2} \cos^2 \frac{u}{2}} = 0$$

$$\cos u \sin u = 0$$

$$\sin^2 u \cos v - \sin u \cos v \frac{1}{\sin u} = 0$$

$$\sin^3 u \cos v - \sin u \cos v = 0$$

$$\sin u \cos v \left(\sin^2 u - 1 \right) = 0$$

$$\sin u = 0 \Rightarrow u = 0, \pi, 2\pi$$

$$\cos v = 0 \Rightarrow v = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\sin u = \pm 1 \Rightarrow u = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\sin^2 u \sin v - \frac{\sin u \cdot \sin v}{\sin u} = 0$$

$$\sin^3 u \sin v - \sin u \sin v = 0$$

$$\sin u \sin v \left(\sin^2 u - 1 \right) = 0$$

$$\sin u = 0 \Rightarrow u = 0, \pi, 2\pi$$

$$\sin v = 0 \Rightarrow v = 0, \pi, 2\pi$$

$$\sin u = \pm 1 \Rightarrow u = \frac{\pi}{2}, \frac{3\pi}{2}$$

$\cos u \sin u = 0$

$\cos u = 0 \qquad u = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin u = 0 \qquad u = 0, \pi, 2\pi$

u 3. uslovu nemamo v pa nam ovo ne igra nikakvu ulogu

$u = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \rightarrow$ singularne krive.

5. Naći parametarske jed. ravne $x - 3y + 2z - 9 = 0$

Proizvoljna tačka $T(1, 2, z)$

$\Rightarrow 2z = 9 - x + 3y = 9 - 1 + 6 = 14 \Rightarrow z = 7$

$\vec{N}(1, -3, 2)$

$\vec{N} \cdot \vec{u} = 0 \qquad \vec{u}(3, 1, m_1)$

$\vec{N} \cdot \vec{v} = 0 \qquad \vec{v}(1, 3, m_2)$

$\cancel{3} - \cancel{3} + 2m_1 = 0 \qquad m_1 = 0$

$1 - 9 + 2m_2 = 0 \qquad m_2 = 4$

$\Rightarrow \quad \vec{u}(3, 1, 0) \quad \vec{v}(1, 3, 4)$

$$\begin{aligned} x &= 1 + 3u + v \\ y &= 2 + u + 3v \\ z &= 7 + 4v \end{aligned}$$

$\qquad T \qquad \vec{u} \qquad \vec{v}$

parametarske jed.

TANGENTNA RAVAN

1. Naći tangentnu ravan i normalu plohe:

$x = u\cos v$

$y = u\sin v$

$z = av$

$u, v \in \mathbb{R}$ u proizvoljnoj tački M_0.

$\vec{r} = u\cos v\,\vec{i} + u\sin v\,\vec{j} + av\,\vec{k}$

$\frac{\partial x}{\partial u} = \cos v$ $\qquad$ $\frac{\partial x}{\partial v} = -u\sin v$

$\frac{\partial y}{\partial u} = \sin v$ $\qquad$ $\frac{\partial y}{\partial v} = u\cos v$

$\frac{\partial z}{\partial u} = 0$ $\qquad$ $\frac{\partial z}{\partial v} = a$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & a \end{vmatrix} = \vec{i}(a\sin v) - \vec{j}(a\cos v) + u\vec{k}$$

tangentna ravan:

$$a\sin v\,(x - u\cos v) - a\cos v\,(y - u\sin v) + u\,(z - av) = 0$$

ili:
$$\begin{vmatrix} x - u\cos v & y - u\sin v & z - av \\ \cos v & \sin v & 0 \\ -u\sin v & -u\cos v & a \end{vmatrix}$$

normala: $\frac{x - a\cos v}{a\sin v} = \frac{y - u\sin v}{-a\cos v} = \frac{z - av}{a}$

2.) Napisati jed. tangentne ravni na plohu:

$x = 2u - v$

$y = u^2 + v^2$

$z = u^3 - v^3$

$u, v \in \mathbb{R}$; u tački $M(3,5,7)$.

prvo naći u i v: $\left.\begin{array}{l} 2u - v = 3 \\ u^2 + v^2 = 5 \\ u^3 - v^3 = 7 \end{array}\right\}$ sis. i uvrstimo u determinantu

Rj. $18x + 3y - 4z - 41 = 0$

3.) Napisati jed. tang. ravni i normale na plohu:

$x = u + v$

$y = u - v$

$z = uv$

u tački za $u = 2$ i $v = 1$.

uvrstimo u i v $\Rightarrow$ $x = 3$, $y = 1$, $z = 2$

Rj. $3x - y - 2z - 4 = 0$ tang. ravan

$\frac{x-3}{3} = \frac{y-1}{-1} = \frac{z-2}{-2}$ normala

4. Zadana je ploha $\vec{r} = \{u\cos v, u\sin v, au\}$ $u \in R$
$v \in [0, 2\pi]$

a) Napisati jed. tangentne ravni i normale u proizvoljnoj tački (u_0, v_0)

b) Napisati jed. tangentne ravni i normale na krivu ~~a=2~~ u tački $u=2$, $v=\frac{\pi}{4}$.

$x = u\cos v$ $\quad y = u\sin v$ $\quad z = au$

$\frac{\partial x}{\partial u} = \cos v$ $\quad \frac{\partial y}{\partial u} = \sin v$ $\quad \frac{\partial z}{\partial u} = a$

$\frac{\partial x}{\partial v} = -u\sin v$ $\quad \frac{\partial y}{\partial v} = u\cos v$ $\quad \frac{\partial z}{\partial v} = 0$

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ \cos v_0 & \sin v_0 & a \\ -u_0\sin v_0 & u_0\cos v_0 & 0 \end{vmatrix} = 0$$

$(x-x_0)\cdot(-au_0\cos v_0) - (y-y_0)\cdot au_0\sin v_0 + (z-z_0)\, u_0(\cos^2 v_0 + \sin^2 v_0) =$

$-au_0\cos v_0(x - u_0\cos v_0) - au_0\sin v_0(y - u_0\sin v_0) + u_0(z - au_0) = 0$

$-au_0\cos v_0\, x + au_0^2\cos^2 v_0 - au_0\sin v_0\, y + au_0^2\sin^2 v_0 + u_0 z - au_0^2 = 0$ $\quad /:u_0$

$-a\cos v_0\, x + \cancel{au_0} - a\sin v_0\, y + z - \cancel{au_0} = 0$

$ax\cos v_0 + ay\sin v_0 - z = 0$ tang. ravan

$$\frac{x - u_0\cos v_0}{-au_0\cos v_0} = \frac{y - u_0\sin v_0}{-au_0\sin v_0} = \frac{z - au_0}{u_0} \quad /:(-u_0)$$

$$\frac{x - u_0\cos v_0}{a\cos v_0} = \frac{y - u_0\sin v_0}{a\sin v_0} = \frac{z - au_0}{-1} \quad /\cdot a$$

$$\frac{x - \cancel{u_0\cos v_0}}{\cos v_0} = \frac{y - u_0\sin v_0}{\sin v_0} = -a(z - au_0)$$

b) $u = 2, \quad v = \frac{\pi}{4}$

$$a \cdot x \cos\frac{\pi}{4} + a y \sin\frac{\pi}{4} - z = 0$$

$$\frac{a\sqrt{2}}{2} x + \frac{a\sqrt{2}}{2} y - z = 0 \quad \Big/ : \frac{a\sqrt{2}}{2}$$

$$x + y - \frac{z}{\frac{a\sqrt{2}}{2}} = 0 \Rightarrow x + y - \frac{2z}{a\sqrt{2}} = 0$$

$$\frac{x - \sqrt{2}}{\frac{a\sqrt{2}}{2}} = \frac{y - \sqrt{2}}{\frac{a\sqrt{2}}{2}} = \frac{z - 2a}{-1} \quad \Big/ \cdot \frac{a\sqrt{2}}{2}$$

$$\frac{x - \sqrt{2}}{1} = \frac{y - \sqrt{2}}{1} = -a\frac{\sqrt{2}}{2}(z - 2a)$$

I DIFERENCIJALNA FORMA

1. Data je sfera svojom parametrizacijom:

$\vec{r} = \{r\sin u\cos v, r\sin u\sin v, r\cos u\}$

$u \in [0, \pi]$, $v \in [-\pi, \pi]$.

a) Naći I. dif. formu pridruženu toj parametrizaciji

b) Naći tang. ravan u tačkama za koje je $u = 0$ i $u = \pi$.

a) $I = E\,du^2 + 2F\,du\,dv + G\,dv^2$

$E = \vec{r}_{uu} = \left(\frac{\partial^2 x}{\partial u^2}, \frac{\partial^2 y}{\partial u^2}, \frac{\partial^2 z}{\partial u^2}\right)$ $\quad \cancel{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$

$F = \vec{r}_{uv} = \left(\frac{\partial^2 x}{\partial u \partial v}, \frac{\partial^2 y}{\partial u \partial v}, \frac{\partial^2 z}{\partial u \partial v}\right)$

$G = \vec{r}_{vv} = \left(\frac{\partial^2 x}{\partial v^2}, \frac{\partial^2 y}{\partial v^2}, \frac{\partial^2 z}{\partial v^2}\right)$

$x = r\sin u\cos v \qquad y = r\sin u\sin v \qquad z = r\cos u$

$\frac{\partial x}{\partial u} = r\cos u\cos v \qquad \frac{\partial y}{\partial u} = r\cos u\sin v \qquad \frac{\partial z}{\partial u} = -r\sin u$

$\frac{\partial x}{\partial v} = -r\sin u\sin v \qquad \frac{\partial y}{\partial v} = r\sin u\cos v \qquad \frac{\partial z}{\partial v} = 0$

$\left(\frac{\partial x}{\partial x \partial y}\right)^2 = \frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} = F$

$E = r^2\cos^2 u\cos^2 v + r^2\cos^2 u\sin^2 v + r^2\sin^2 u =$

$= r^2\cos^2 u + r^2\sin^2 v = r^2$

$G = r^2\sin^2 u\sin^2 v + r^2\sin^2 u\cos^2 v + 0 = r^2\sin^2 u$

$F = -r^2 \sin u \sin v \cos u \cos v + r^2 \sin u \sin v \cos u \cos v = 0$

$$\boxed{I = ds^2 = r^2 du^2 + r^2 \sin^2 u \, dv^2}$$

b) $u = 0$; $u = \pi$;

$$\begin{vmatrix} x - r\sin u \cos v & y - r \sin u \sin v & z - r\cos u \\ r\cos u \cos v & r\cos u \sin v & -r\sin u \\ -r\sin u \sin v & r \sin u \cos v & 0 \end{vmatrix} = 0$$

$(x - r\sin u \cos v) \cdot r^2 \sin^2 u \cos v + (y - r\sin u \sin v)\, r^2 \sin^2 u \sin v +$
$+ (z - r\cos u)(r^2 \sin u \cos^2 v \cos u + r^2 \sin u \sin^2 v \cos u) = 0$

$x r^2 \sin^2 u \cos v - r^3 \sin^3 u \cos^2 v + y r^2 \sin^2 u \sin v - r^3 \sin^3 u \sin^2 v +$
$+ z r^2 \sin u \cos u - r^3 \sin u \cos^2 u = 0 \quad / : r^2 \sin u$

$x \sin u \cos v - r \sin^2 u + y \sin u \sin v + z \cos u - r \cos^2 u = 0$

$u = 0 \qquad z \cdot \underbrace{\cos 0^\circ}_{1} - r \cdot \underbrace{\cos^2 0^\circ}_{1} = 0$

$z - r = 0 \Rightarrow z = r$

$u = \pi \qquad z \cdot \cos \pi - r \cdot \cos^2 \pi = 0$

$z \cdot (-1) - r \cdot (-1) \cdot (-1) = 0$

$-z - r = 0 \Rightarrow -z = r \Rightarrow z = -r$

2. Naći površinu četverougla na helikoidu zadanom:

$x = au\cos v$

$y = au\sin v$

$z = bv$

$u, v \in \mathbb{R}$, koji je omeđen sa $u=0$, $u=\frac{b}{a}$, $v=0$, $v=1$.

$$P = \iint\limits_{(K)} \sqrt{EG-F^2}\, du\, dv$$

$\frac{\partial x}{\partial u} = a\cos v \qquad \frac{\partial y}{\partial u} = a\sin v \qquad \frac{\partial z}{\partial u} = 0$

$\frac{\partial x}{\partial v} = -au\sin v \qquad \frac{\partial y}{\partial v} = au\cos v \qquad \frac{\partial z}{\partial v} = b$

$E = a^2\cos^2 v + a^2\sin^2 v + 0 = a^2$

$G = a^2u^2\sin^2 v + a^2u^2\cos^2 v + b^2 = a^2u^2 + b^2$

$F = -a^2u\sin v\cos v + a^2u\sin v\cos v + 0 = 0$

$E\cdot G = a^4u^2 + a^2b^2$

$E\cdot G - F^2 = a^4u^2 + a^2b^2$

$$P = \int_0^{\frac{b}{a}} du \int_0^1 \sqrt{a^4u^2+a^2b^2}\, dv = \int_0^{\frac{b}{a}} du \int_0^1 a\sqrt{a^2u^2+b^2}\, dv =$$

$$= \int_0^{\frac{b}{a}} a\sqrt{a^2u^2+b^2}\cdot v\Big|_0^1 du = \int_0^{\frac{b}{a}} a\sqrt{a^2u^2+b^2}\, du = \int_0^{\frac{b}{a}} \frac{a(a^2u^2+b^2)}{\sqrt{a^2u^2+b^2}}\, du =$$

$$= a\cdot\underbrace{\int \frac{a^2u^2}{\sqrt{a^2u^2+b^2}}\, du}_{I_2} + a\underbrace{\int \frac{b^2}{\sqrt{a^2u^2+b^2}}\, du}_{I_1}$$

$$I_1 = \left| au = bt \right| = \frac{1}{a}\ln\frac{au+\sqrt{a^2u^2+b^2}}{b}$$

$a^2u^2+b^2 = t$

$$I_2 = \left| \begin{matrix} x = u & dv = \frac{a^2u\, du}{\sqrt{a^2u^2+b^2}} \\ dx = du & v = \frac{1}{2}\int \frac{2a^2u\, du}{\sqrt{a^2u^2+b^2}} = \cdot\frac{1}{2}\cdot 2\sqrt{a^2u^2+b^2} = \sqrt{a^2u^2+b^2} \end{matrix} \right|$$

$au = b\sin t$

$a\,du = b\cos t\,dt$

$$a\int_0^{\pi/2} \sqrt{\sin^2 t + 1}\cdot b\cos t\,dt = ab\int_0^{\frac{\pi}{2}} \cos^2 t\,dt =$$

$$I_2 = u\sqrt{a^2u^2+b^2} - \int \sqrt{a^2u^2+b^2}\,du = \left| \begin{array}{ll} au = bt & du = \frac{b}{a}dt \\ a\,du = b\,dt & \end{array} \right|$$

$$= u\sqrt{a^2u^2+b^2} - \int \underbrace{\sqrt{b^2t^2+b^2}}_{b\sqrt{t^2+1}}\cdot\frac{b}{a}\,dt = u\sqrt{a^2u^2+b^2} - \int \frac{b^2}{a}\sqrt{t^2+1}\,dt$$

$$S = \frac{b^2}{2}\ln\frac{au+\sqrt{a^2u^2+b^2}}{b} + \frac{au}{2}\sqrt{a^2u^2+b^2}\Bigg]\Bigg|_0^{\frac{b}{a}} = \frac{b^2}{2}\left[\ln\left(1+\sqrt{2}\right)+\sqrt{2}\right]$$